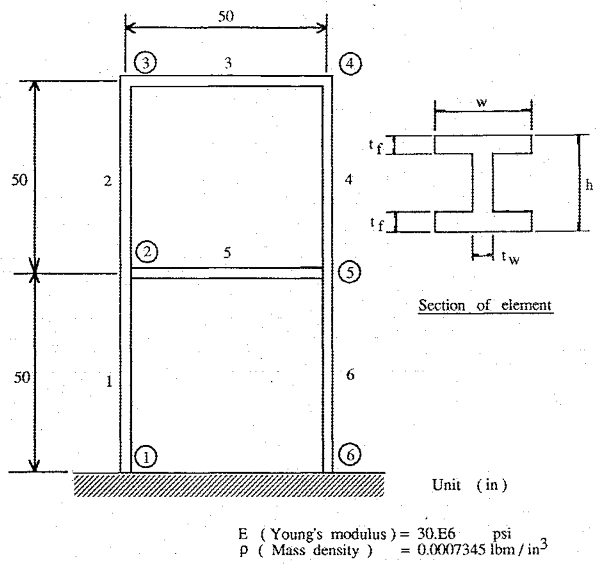

Fig. 2 Plane-frame problem.

as compared with pure divided-difference schemes. The maximum number of iterations required for an error tolerance of 10^{-7} is five, which indicates rapid convergence. Also, the algorithm does not require computation of all eigenvectors to find the sensitivity of a few specific eigenvectors.

Summary and Conclusions

A numerical method has been presented for design sensitivity analysis. The idea is based on using iterative methods for reanalysis of the structure due to a small perturbation in the design variable. A forward-difference scheme then yields the approximate sensitivity. Algorithms for displacement and stress sensitivity as well as for eigenvalues and eigenvector sensitivity are developed. The iterative schemes have been modified so that the coefficient matrices are constant and hence decomposed only once. The convergence is found to be very rapid. Implementation of the algorithms is found to be simple. The method can extend to eigenvalue sensitivity of problems with repeated roots. This extension is also important to avoid ill conditioning of the coefficient matrix in the vicinity of bifurcation points, which occurs when nonlinear structural response is considered.

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Higher-Order Finite Element for Short Beams

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Introduction

PROBLEMS involving relatively short beams are common in some aerospace structures. In such beams, the transverse shear has an important effect on the deformations and stresses. The elementary Bernoulli-Euler beam theory, which does not include shear deformation, is inadequate. An early theory that includes transverse shear was presented by Timoshenko[1] in 1921. This theory assumed that plane sections remain plane during the deformation but not necessarily normal to the middle surface. Since then investigators have attempted improvement by modifying the shear constant in Timoshenko's theory.[2] More recently, a number of papers have appeared which offer alternate approaches. In 1981, Levinson[3] introduced a new beam theory that included warping of the cross section and imposed a shear-free condition on the top and bottom surfaces. Rehfield and Murthy,[4] in 1982, solved a simply supported beam problem using a modified plane stress elasticity solution to include transverse shear and normal strains. Their approach differs from Timoshenko's because of assumptions regarding the stresses rather than the strains. In 1984, Murty[5] presented another higher-order beam theory. In 1986, Suzuki[6] presented a theory for short beams also, using assumptions about the stress distribution. Some articles have appeared which utilize finite elements with the shear effect included.[7-12] Most of these removed the restriction that cross sections normal to the undeformed middle surface of the beam remain normal during the deformation (similar to the Timoshenko theory). These normal cross sections, however, do remain plane. The finite element presented here removes this restriction by allowing the cross sections to deform into a shape that can be described by a function that includes quadratic and cubic terms as well as the linear one. Although these additional terms create a more complex element (six more degrees of freedom), the resulting element is shown to provide good results for both stresses and displacements when compared with other theories for short beams. Finite elements are, of course, easily extrapolated to practical problems with more complex loading and boundary conditions.

Formulation

The proposed beam finite element is shown in Fig. 1a. It has length ℓ and a cross section symmetric about the y axis. The x and y displacements of a typical point in the beam are u and v, respectively. The x and y displacements of the corresponding point on the middle surface of the beam are U and V. The rotation of the cross section is characterized by the three terms, Φ_1, Φ_2, and Φ_3. The expressions for u and v in terms of the five functions U, V, Φ_1, Φ_2, and Φ_3 are

$$u = U - \Phi_1 y - \Phi_2 y^2 - \Phi_3 y^3, \quad v = V \tag{1}$$

where U, V, Φ_1, Φ_2, and Φ_3 are functions of x (the shape functions) to be described below in Eqs. (4). The axial strain ϵ_x and the shear strain γ_{xy} are then

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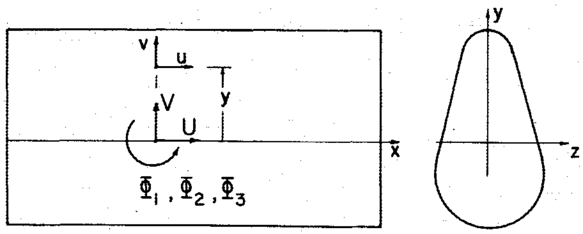

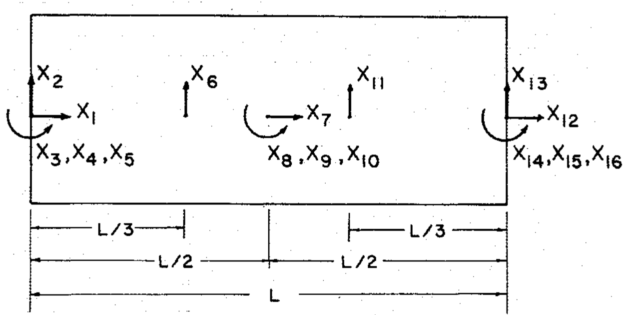

Fig. 1 Displacements and nodal variables for beam finite element.

$$\epsilon_x = U' - \Phi_1' y - \Phi_2' y^2 - \Phi_3' y^3$$

$$\gamma_{xy} = V' - \Phi_1 - 2\Phi_2 y - 3\Phi_3 y^2 \tag{2}$$

where primes denote derivatives with respect to x. The corresponding axial stress σ_x and shear stress τ_{xy} are

$$\sigma_x = E\epsilon_x, \quad \tau_{xy} = G\gamma_{xy} \tag{3}$$

where E and G are Young's modulus and the shear modulus respectively.

In previous finite elements modeling beams with shear effect, the terms Φ_2 and Φ_3 have not been included. As a consequence, these elements force cross sections that were originally normal to the undeformed middle surface of the beam to remain plane during the deformation of the beam. (They undergo rigid body motion.) Including these two additional terms in the present element allows the cross sections additional freedom, and hence the element becomes considerably more flexible. The shape function for the lateral displacement V is chosen as a cubic in x. For consistency of deformation, this requires that the other four shape functions must all be quadratic in x. Since the axial displacement of a point not on the middle surface is a linear function of U, Φ_1, Φ_2, and Φ_3, the degree of polynomial used for all four of these functions must be the same. In addition, the shear strain γ_{xy} is a linear function of these four functions as well as of dV/dx. Consequently, the degree of polynomial used for V must be one order higher than those used for the other four functions in order to insure compatibility. Accordingly, let

$$U = C_1 + C_2 x + C_3 x^2, \quad V = C_4 + C_5 x + C_6 x^2 + C_7 x^3$$

$$\Phi_1 = C_8 + C_9 x + C_{10} x^2, \quad \Phi_2 = C_{211} + C_{12} x + C_{13} x^2,$$

$$\Phi_3 = C_{14} + C_{15} x + C_{16} x^2 \tag{4}$$

The nodal variables are shown in Fig. 1b. The element has five nodes with 16 degrees of freedom: five at each of the ends, one at each of the third points, and four at the midpoint. Sixteen degrees of freedom, although large for a beam element, allow sufficient flexibility for modeling the behavior of a relatively short beam. It is to be noted that the three axial and three quadratic rotational degrees of freedom can be discarded in beams that are symmetric about the z axis (hence reducing the number of degrees of freedom to 10). However, beam elements are often used to model eccentric stiffeners on plates or shells, and in such cases the additional six terms are necessary. Accordingly, they are included here for completeness.

Following the usual derivation process for displacement-type finite elements, the relation between the nodal load and displacement vectors is obtained as

$$F = KX \tag{5}$$

where the terms in the 16×16 stiffness matrix K contain the usual material and geometric properties and, in addition, some higher-order area integrals of the following form (with n ranging from 0 to 6):

$$A_n = \int_{\text{area}} y^n \, \text{darea} \tag{6}$$

The parameter A_0 is, of course, the area of the cross section. The parameter A_1 is zero, since the axes y,z were selected at the centroid. Note that A_0 and A_2 (the area and the second moment of the area of the cross section) are the only two cross-sectional properties needed in the elementary theory or in the Timoshenko theory. This new theory, in contrast, requires six properties.

Examples

The first example is a cantilever beam (length $= L$) of rectangular cross section (height $= H$) with a lateral force at the end. The selected values of the parameters are

$$G/E = 0.377, \quad H/L = 0.2 \tag{7}$$

The results for the deflection at the free end, together with the maximum normal stress, are given in Table 1 normalized with respect to values obtained from Bernoulli-Euler beam theory. Results from Timoshenko[1] beam theory by Murty[5] and Levinson[3] are given for comparison. Additional data, with the shear

Table 1 Maximum deflection and bending stress in a cantilever beam with end load

	δ/δ_{B-E}	$\sigma_{max}/\sigma_{max_{B-E}}$
$G/E = 0.377$		
Timoshenko[1]	1.03	1.00
Murty[5]	1.03	1.06
Levinson[3]	1.04	—
Present element	1.03	1.06
$G/E = 0.0377$		
Timoshenko[1]	1.27	1.00
Murty[5]	1.31	1.20
Levinson[3]	1.40	—
Present element	1.31	1.20
$G/E = 0.00377$		
Timoshenko[1]	3.66	1.00
Murty[5]	4.04	1.67
Levinson[3]	4.98	—
Present element	4.04	1.64

Table 2 Maximum deflection of a simply supported beam with a uniform load δ/δ_{B-E}

H/L	1.0	0.8	0.6	0.4	0.2	0.1
Timoshenko[1]	4.12	3.12	2.12	1.50	1.12	1.03
Levinson[3]	3.50	2.60	1.90	1.40	1.10	1.03
Rehfield and Murthy[4]	3.28	2.46	1.82	1.36	1.09	1.02
Present element	3.44	2.57	1.89	1.40	1.10	1.03

modulus reduced by a factor of 10 and then 100 (to approximate a laminated sandwich beam with a soft core), are also presented in Table 1. Results from the present theory (using 20 elements) agree well with those of Murty. The Timoshenko beam theory agrees reasonably well as regards deflection. The Levinson theory gives a larger deflection than the others. Both the Bernoulli-Euler and Timoshenko beam theories give values for the maximum normal stress substantially lower than those obtained by the present theory or by Murty.

The second example is a simply supported beam of rectangular cross section with uniform load. The value of Poisson's ratio is 0.3. The height-to-length ratio (H/L) is varied from 0.1 to 1.0. The latter is a very short beam. Deflection results normalized to those of Bernoulli-Euler theory are presented in Table 2. The present theory is compared with those from Timoshenko,[1] Levinson,[3] and Rehfield and Murthy.[4] A shear correction factor of $k = 5/6$ has been incorporated in the Timoshenko results. The present theory and the Levinson theory agree closely and are slightly more flexible than the Rehfield and Murthy theory. The Timoshenko theory (with the shear correction factor) yields the largest deflection for this case.

Conclusions

The higher-order finite element presented here gives results for short beams which are in close agreement with most recent theories. Inclusion of the terms that permit warping of the cross section makes a substantial difference in the value of the maximum bending stress in such structural components.

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